NOTES DIFFERENTIAL CALCULUS

THE CONCEPT OF A LIMIT

EXAMPLE 1

Determine the following limits:

(a) $\lim_{x\to 1}(2x^2+4)$

$=(2(1)^2+4)$

$=6$

(b) $\lim_{x\to 2}\frac{x^2-4}{x+1}$

$=\frac{(2)^2-4}{2+1}$

$=0$

(c) $\lim_{x\to 1}\frac{x^2-1}{x-1}$

In this example, it is first necessary to simplify the expression before determining the limit, as the denominator will be zero if $x=1$ and division by zero is undefined.

$\therefore \lim_{x\to 1}\frac{x^2-1}{x-1}$

$=\lim_{x\to 1}\frac{(x+1)(x-1)}{(x-1)}$

$=\lim_{x\to 1}(x+1)$

$=1+1=2$

Remember the following factorisation principles:

$a^2-b^2=(a+b)(a-b)$	$(b-a)=-(a-b)$
$a^2-2ab+b^2=(a-b)^2$	$a^2+2ab+b^2=(a+b)^2$
$a^3-b^3=(a-b)(a^2+ab+b^2)$	$a^3+b^3=(a+b)(a^2-ab+b^2)$

RULES OF DIFFERENTIATION

The following rules of differentiation are applicable for determining the gradient of a function given that the value of k is a constant.

Rule 1 If $f(x) = kx^n$, then $f'(x) = k.nx^{n-1}$

Example:

If $f(x) = 8x^3$, then $f'(x) = 8 \times 3x^{3-1} = 24x^2$

Rule 2 If $f(x) = kx$, then $f'(x) = k$

Example:

If $f(x) = 9x$, then $f'(x) = 9$

Rule 3 If $f(x) = k$, then $f'(x) = 0$

Example:

If $f(x) = 9$, then $f'(x) = 0$

If $f(x) = p$ (where p is a constant), then $f'(x) = 0$

Rule 1: Multiply out before you differentiate.

EXAMPLE 11

Determine the following:

(a) $f'(x)$ if $f(x) = (4x-3)^2$

Here we first need to expand and simplify so as to remove the brackets.

$f(x) = (4x-3)^2$

$\therefore f(x) = 16x^2 - 24x + 9$

$\therefore f'(x) = 16 \times 2x^{2-1} - 24 + 0$

$\therefore f'(x) = 32x - 24$

Rule 2: To get rid of the monomial in denominator ,write each term over the denominator and simplify:

(c) $D_x\left[\frac{x^2-x-6}{2x^2}\right]$

$= D_x\left[\frac{x^2}{2x^2}-\frac{x}{2x^2}-\frac{6}{2x^2}\right]$

$= D_x\left[\frac{1}{2}-\frac{1}{2x}-\frac{3}{x^2}\right]$

$= D_x\left[\frac{1}{2}-\frac{1}{2}x^{-1}-3x^{-2}\right]$

$= 0-\frac{1}{2}(-1)x^{-1-1}-3(-2)x^{-2-1}$

$= \frac{1}{2}x^{-2}+6x^{-3}$

$= \frac{1}{2x^2}+\frac{6}{x^3}$

Rule 3: To get rid of the binomial in the denominator, factorise, simplify then differentiate.

(d) $D_x\left[\frac{3x^3-7x^2-6x}{3-x}\right]$

$= D_x\left[\frac{x(3x^2-7x-6)}{3-x}\right]$

$= D_x\left[\frac{x(3x+2)(x-3)}{3-x}\right]$

$= D_x\left[\frac{x(3x+2)(x-3)}{-(x-3)}\right]$

$= D_x\left[-x(3x+2)\right]$

$= D_x\left[-3x^2-2x\right]$

$= -6x-2$

RULE 4: Get rid of square and cube roots

(b) $\frac{dy}{dx}$ if $y = \frac{1}{2\sqrt[4]{x^3}}$

Here we need to first get rid of the divide line over the variable as well as the root sign.

$y = \frac{1}{2\sqrt[4]{x^3}}$

$\therefore y = \frac{1}{2x^{\frac{3}{4}}}$ ($\sqrt[n]{a^m} = a^{\frac{m}{n}}$)

$\therefore y = \frac{1}{2}x^{-\frac{3}{4}}$ (Exponential definition (c))

$\therefore \frac{dy}{dx} = \frac{1}{2} \times -\frac{3}{4}x^{-\frac{3}{4}-1}$ (Rule 1)

$\therefore \frac{dy}{dx} = -\frac{3}{8}x^{-\frac{7}{4}}$

$\therefore \frac{dy}{dx} = -\frac{3}{8x^{\frac{7}{4}}}$

$\therefore \frac{dy}{dx} = -\frac{3}{8\sqrt[4]{x^7}}$

Determine:

(a) $f'(x)$ if $f(x) = 3x^6$	(a) $\frac{dy}{dx}$ if $y = 3x^6$	(a) $D_x\left[3x^6\right]$
Rule 1: $f(x) = 3x^6$ $\therefore f'(x) = 3 \times 6x^{6-1}$ $\therefore f'(x) = 18x^5$	Rule 1: $y = 3x^6$ $\therefore \frac{dy}{dx} = 3 \times 6x^{6-1}$ $\therefore \frac{dy}{dx} = 18x^5$	Rule 1: $D_x\left[3x^6\right]$ $= 3 \times 6x^{6-1}$ $= 18x^5$
(b) $f'(x)$ if $f(x) = 10x$	(b) $\frac{dy}{dx}$ if $y = 10x$	(b) $D_x\left[10x\right]$
Rule 2: $f(x) = 10x$ $\therefore f'(x) = 10$	Rule 2: $y = 10x$ $\therefore \frac{dy}{dx} = 10$	Rule 2: $D_x\left[10x\right]$ $= 10$
(c) $f'(x)$ if $f(x) = -8$	(c) $\frac{dy}{dx}$ if $y = -8$	(c) $D_x\left[-8\right]$
Rule 3: $f(x) = -8$ $\therefore f'(x) = 0$	Rule 3: $y = -8$ $\therefore \frac{dy}{dx} = 0$	Rule 3: $D_x\left[-8\right]$ $= 0$
(d) $f'(x)$ if $f(x) = m$ (m is a constant)	(d) $\frac{dy}{dx}$ if $y = m$ (m is a constant)	(d) $D_x\left[m\right]$ (m is a constant)
Rule 3: $f(x) = m$ $\therefore f'(x) = 0$	Rule 3: $y = m$ $\therefore \frac{dy}{dx} = 0$	Rule 3: $D_x\left[m\right]$ $= 0$
(e) $g'(x)$ if $g(x) = \frac{x^3}{3}$	(e) $\frac{dy}{dx}$ if $y = \frac{x^3}{3}$	(e) $D_x\left[\frac{x^3}{3}\right]$
$g(x) = \frac{x^3}{3}$ First use exp def (a): $\therefore g(x) = \frac{1}{3}x^3$ Now use Rule 1: $\therefore g'(x) = \frac{1}{3} \times 3x^{3-1}$ $\therefore g'(x) = x^2$	$y = \frac{x^3}{3}$ First use exp def (a): $\therefore y = \frac{1}{3}x^3$ Now use Rule 1: $\therefore \frac{dy}{dx} = \frac{1}{3} \times 3x^{3-1}$ $\therefore \frac{dy}{dx} = x^2$	$D_x\left[\frac{x^3}{3}\right]$ $= D_x\left[\frac{1}{3}x^3\right]$ exp def (a) $= \frac{1}{3} \times 3x^{3-1}$ Rule 1 $= x^2$

EXAMPLE 5

Find, from first principles, the gradient of $f(x) = 3x$ at any point.

Solution

We know that the graph of $y = 3x$ is a straight line with a gradient of 3.
By using first principles, we can verify this fact.

Step 1 Write down the formula for finding gradient from first principles:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2 Write down the given $f(x)$ and then determine $f(x+h)$:

$f(x) = 3x$ $\qquad \therefore f(x+h) = 3(x+h)$

$\therefore f(x+h) = 3x + 3h$

Step 3 Substitute the expressions for $f(x)$ and $f(x+h)$ into the formula and then simplify the expression and evaluate the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{3x + 3h - 3x}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{3h}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} 3$$

$$\therefore f'(x) = 3$$

Hence the gradient of the line is 3.

EXAMPLE 7

(a) Find, from first principles, the gradient of $f(x) = 1 - 3x^2$ at any point.

(b) Hence find $f'(-4)$, the derivative of f at $x = -4$.

(c) What is the gradient of the tangent to f at $x = 5$?

Solution

(a) **Step 1**

Write down the formula for finding gradient from first principles:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Step 2

Write down the given $f(x)$ and then determine $f(x+h)$:

$f(x) = 1 - 3x^2$

$$\therefore f(x+h) = 1 - 3(x+h)^2$$
$$\therefore f(x+h) = 1 - 3(x^2 + 2xh + h^2)$$
$$\therefore f(x+h) = 1 - 3x^2 - 6xh - 3h^2$$

Step 3

Substitute the expressions for $f(x)$ and $f(x+h)$ into the formula and then simplify the expression and evaluate the limit:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$\therefore f'(x) = \lim_{h \to 0} \frac{(1 - 3x^2 - 6xh - 3h^2) - (1 - 3x^2)}{h}$$
$$\therefore f'(x) = \lim_{h \to 0} \frac{1 - 3x^2 - 6xh - 3h^2 - 1 + 3x^2}{h}$$
$$\therefore f'(x) = \lim_{h \to 0} \frac{-6xh - 3h^2}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{h(-6x - 3h)}{h}$$
$$\therefore f'(x) = \lim_{h \to 0} (-6x - 3h)$$
$$\therefore f'(x) = -6x - 3(0)$$
$$\therefore f'(x) = -6x$$

(b) Since $f'(x) = -6x$ represents the gradient of the graph at any point on the graph, it is now easy to determine the gradient (derivative) at $x = -4$:

$f'(x) = -6x$

$\therefore f'(-4) = -6(-4) = 24$

(c) The gradient of the tangent to the graph at $x = 5$ can now also be determined:

$f'(x) = -6x$

$\therefore f'(5) = -6(5) = -30$